

On Murty-Simon Conjecture

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Abstract

A graph is diameter two edge-critical if its diameter is two and the deletion of any edge increases the diameter. Murty and Simon conjectured that the number of edges in a diameter two edge-critical graph on n vertices is at most $\lfloor \frac{n^2}{4} \rfloor$ and the extremal graph is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. In the series papers [8–10], the Murty-Simon Conjecture stated by Haynes et al. is not the original conjecture, indeed, it is only for the diameter two edge-critical graphs of even order. Haynes et al. proved the conjecture for the graphs whose complements have diameter three but only with even vertices. In this paper, we prove the Murty-Simon Conjecture for the graphs whose complements have diameter three, not only with even vertices but also odd ones.

1 Introduction

All graphs considered in this paper are simple. We adopt notation and terminology commonly used in the literature. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex v in a graph G , denoted by $N_G(v)$, is the set of all the vertices adjacent to the vertex v , i.e., $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* of a vertex v in G , denoted by $N_G[v]$, is defined by $N_G[v] = N_G(v) \cup \{v\}$. For a subset $S \subseteq V$, the *neighborhood of the set* S in G is the set of all vertices adjacent to vertices in S , this set is denoted by $N_G(S)$, and the *closed neighborhood of S* by $N_G[S] = N_G(S) \cup S$. Let S and T be two subsets (not necessarily disjoint) of $V(G)$, $[S, T]$ denotes the set of edges of G with one end in S and the other in T , and $e_G(S, T) = |[S, T]|$. If every vertex in S is adjacent to each vertex in T , then we say that $[S, T]$ is full. If $S \subseteq V(G)$, and u, v are two nonadjacent vertices in G , then we say that xy is a *missing edge* in S (rather than “ uv is a missing edge in $G[S]$ ”).

The *complement* G^c of a simple graph $G = (V, E)$ is the simple graph with vertex set V , two vertices are adjacent in G^c if and only if they are not adjacent in G .

Given a graph G and two vertices u and v in it, the *distance* between u and v in G , denoted by $d_G(u, v)$, is the length of a shortest u - v path in G ; if there is no path connecting u and v , we define $d_G(u, v) = \infty$. The *diameter* of a graph G , denoted by $\text{diam}(G)$, is the maximum distance between any two vertices of G . Clearly, $\text{diam}(G) = \infty$ if and only if G is disconnected.

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A subset $S \subseteq V$ is called a *dominating set* (**DS**) of a graph G if every vertex $v \in V$ is an element of S or is adjacent to a vertex in S , that is, $N_G[S] = V$. The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G .

A subset $S \subseteq V$ is a *total dominating set*, abbreviated **TDS**, of G if every vertex in V is adjacent to a vertex in S , that is $N_G(S) = V$. Every graph without isolated vertices has a TDS, since V is a trivial TDS. The *total domination number* of a graph G , denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS in G . For the graph with isolated vertices, we define its total domination number to be ∞ . Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2].

For two vertex subsets X and Y , we say that X *dominates* Y (*totally dominates* Y , respectively) if $Y \subseteq N_G[X]$ ($Y \subseteq N_G(X)$, respectively); sometimes, we also say that Y is *dominated* by X (*totally dominated* by X , respectively).

For three vertices $u, v, w \in V(G)$, the symbol $uv \rightarrow w$ means that $\{u, v\}$ dominates $G - w$, but $uw \notin E(G)$, $vw \notin E(G)$ and $uv \in E(G)$.

A graph G is said to be *diameter- d edge-critical* if $\text{diam}(G) = d$ and $\text{diam}(G - e) > \text{diam}(G)$ for any edge $e \in E(G)$. Gliviak [5] proved the impossibility of characterization of diameter- d edge-critical graphs by finite extension or by forbidden subgraphs. Plesník [11] observed that all known minimal graphs of diameter two on n vertices have no more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges. Independently, Murty and Simon (see [1]) conjectured the following:

Murty-Simon Conjecture. If G is a diameter-2 edge-critical graph on n vertices, then $|E(G)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. Moreover, equality holds if and only if G is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Let G be a diameter-2 edge-critical graph on n vertices. Plesník [11] proved that $|E(G)| < 3n(n - 1)/8$. Caccetta and Häggkvist [1] obtained that $|E(G)| < 0.27n^2$. Fan [3] proved the first part of the Murty-Simon Conjecture for $n \leq 24$ and for $n = 26$; and

$$|E(G)| < \frac{1}{4}n^2 + (n^2 - 16.2n + 56)/320 < 0.2532n^2$$

for $n \geq 25$. Füredi [4] proved the Murty-Simon Conjecture for $n > n_0$, where n_0 is not larger than a tower of 2's of height about 10^{14} .

A graph is *total domination edge critical* if the addition of any edge decrease the total domination number. If G is total domination edge critical with $\gamma_t(G) = k$, then we say that G is a k - γ_t -*edge critical graph*. Haynes et al. [7] proved that the addition of an edge to a graph without isolated vertices can decrease the total domination number by at most two. A graph G with the property that $\gamma_t(G) = k$ and $\gamma_t(G + e) = k - 2$ for every missing edge e in G is called a k -*supercritical* graph.

Theorem 1.1 (Hanson and Wang [6]). A nontrivial graph G is dominated by two adjacent vertices if and only if the diameter of G^c is greater than two.

Corollary 1. A graph G is diameter-2 edge-critical on n vertices if and only if the total domination number of G^c is greater than two but the addition of any edge in G^c decrease the total domination number to be two, that is, G^c is $K_1 \cup K_{n-1}$ or 3- γ_t -edge critical or 4-supercritical.

The complement of G is $K_1 \cup K_{n-1}$ if and only if G is $K_{1,n-1}$. Clearly, the Murty-Simon Conjecture holds for $K_{1,n-1}$.

The 4-supercritical graphs are characterized in [12].

Theorem 1.2. A graph H is 4-supercritical if and only if H is the disjoint union of two nontrivial complete graphs.

The complement of a 4-supercritical graph is a complete bipartite graph. The Murty-Simon Conjecture holds for the graphs whose complements are 4-supercritical, i.e., complete bipartite graphs.

Therefore, we only have to consider the graphs whose complements are $3\gamma_t$ -edge critical.

For $3\gamma_t$ -edge critical graphs, the bound on the diameter is established in [7].

Theorem 1.3. If G is a $3\gamma_t$ -edge critical graph, then $2 \leq \text{diam}(G) \leq 3$.

Hanson and Wang [6] proved the first part of the Murty-Simon Conjecture for the graphs whose complements have diameter three. Recently, Haynes, Henning, van der Merwe and Yeo [8] proved the second part for the graphs whose complements are $3\gamma_t$ -edge critical graphs with diameter three but only with even vertices. Also, Haynes et al. [10] proved the Murty-Simon Conjecture for the graphs of even order whose complements have vertex connectivity ℓ , where $\ell = 1, 2, 3$. Haynes, Henning and Yeo [9] proved the Murty-Simon Conjecture for the graphs whose complements are claw-free.

In this paper, we prove the Murty-Simon Conjecture for the graphs whose complements are $3\gamma_t$ -edge critical graphs with diameter three, not only with even vertices but also odd ones. This theorem includes the result obtained by Haynes et al. [9]. We use the technique developed in [9], and the proof is processed by a series of claims, a few claims are the same with them in [9], but to make the paper self contained, we give a full proof of them.

Let G be a $3\gamma_t$ -edge critical graph. Then the addition of any edge e decrease the total domination number to be two, that is, $G + e$ is dominated by two adjacent vertices x and y ; we call such edge xy *quasi-edge* of e . Note that xy must contain at least one end of e . Clearly, quasi-edge of e may not be unique. If $xy \mapsto w$, then xy is quasi-edge of the missing edge xw , and also quasi-edge of missing edge yw ; conversely, if xy is quasi-edge of a missing edge, then there exists an unique vertex w such that $xy \mapsto w$. So, if $xy \rightarrow w$, we write $\text{un}(xy) = w$.

From the definition of $3\gamma_t$ -edge critical graph, we have the following frequently used observation.

Observation 1. If G is a $3\gamma_t$ -edge critical graph and uv is a missing edge in it, then either

- (i) $\{u, v\}$ dominates G ; or
- (ii) there exists a vertex z such that $uz \mapsto v$ or $zv \mapsto u$.

For notation and terminology not defined here, we refer the reader to [8].

2 Main results

Theorem 2.1. If G is a $3\gamma_t$ -edge critical graph on n vertices with diameter three, then $|E(G^c)| < \left\lfloor \frac{n^2}{4} \right\rfloor$.

Proof. Suppose, to the contrary, that $|E(G^c)| \geq \left\lfloor \frac{n^2}{4} \right\rfloor$. Assume that $d_G(u_0, v_0) = 3$ and $\deg_G(u_0) \leq \deg_G(v_0)$. Let $A = \{v \mid d_G(u_0, v) = 1\}$, $B = \{v \mid d_G(u_0, v) = 2\}$, $C = \{v \mid d_G(u_0, v) = 3\}$. Hence, $\{u_0\}, A, B, C$ is a partition of $V(G)$.

Claim 1. For every missing edge e in A or $B \cup C$, quasi-edges of e are in $[A, B]$. Consequently, C is a clique. Moreover, for every edge in $[A, B]$, it is quasi-edge of at most one missing edge in A or $B \cup C$.

Proof. Suppose that xy is a missing edge in A . Consider $G + xy$, since $\{x, y\}$ does not dominate $\{v_0\}$, there exists a vertex z such that $xz \mapsto y$ or $zy \mapsto x$. In either case, neither x nor y dominate v_0 , so z dominates v_0 , then $z \in N_G[v_0]$, and thus $z \in B$.

Suppose that xy is a missing edge in $B \cup C$. Consider $G + xy$, since $\{x, y\}$ does not dominate $\{u_0\}$, there exists a vertex z such that $xz \mapsto y$ or $zy \mapsto x$. In either case, neither x nor y dominate u_0 , then z dominates u_0 , and thus $z \in N_G(u_0) = A$.

Let uv be an arbitrary edge in $[A, B]$, by Observation 1, it is quasi-edge of at most one missing edge in A or $B \cup C$. \square

Now, we have

$$\left\lfloor \frac{n^2}{4} \right\rfloor \leq |E(G^c)| \leq |A \cup \{u_0\}| \times |B \cup C| \leq \left\lfloor \frac{n^2}{4} \right\rfloor \quad (2.1)$$

Therefore, equalities in (2.1) holds, it implies that

Claim 2. For every missing edge e in A or $B \cup C$, there exists precisely one quasi-edge of e in $[A, B]$; conversely, for every edge in $[A, B]$, it is the quasi-edge of a missing edge in A or $B \cup C$. Moreover, $|B \cup C| = |A| + 1$ or $|A| + 2$.

Claim 3. If $u_1, u_2 \in A$ and $v_1, v_2 \in B$, $\{u_1v_1, u_2v_2\} \subseteq E(G^c)$ and $\{u_1v_2, u_2v_1\} \subseteq E(G)$, then $\{u_1u_2, v_1v_2\} \subseteq E(G)$.

Proof. If $u_1u_2 \notin E(G)$, then both u_1v_2 and u_2v_1 are quasi-edge of u_1u_2 , a contradiction. Similarly, we can prove that $v_1v_2 \in E(G)$. \square

Claim 4. If u_1u_2 is a missing edge in A and $\deg_B(u_1) \geq \deg_B(u_2)$, then $N_B(u_1) = N_B(u_2) \cup \{y\}$, where y is the end (in B) of the quasi-edge of u_1u_2 . Similarly, if v_1v_2 is a missing edge in B and $\deg_A(v_1) \geq \deg_A(v_2)$, then $N_A(v_1) = N_A(v_2) \cup \{x\}$, where x is the end (in A) of the quasi-edge of v_1v_2 . Consequently, the missing edges in A (resp. in B) form a bipartite graph on A (resp. on B).

Proof. Let u_1u_2 be a missing edge in A . Suppose that $N_B(u_1) \not\subseteq N_B(u_2)$ and $N_B(u_2) \not\subseteq N_B(u_1)$. Choose a vertex $v_1 \in N_B(u_2) \setminus N_B(u_1)$ and a vertex v_2 in $N_B(u_1) \setminus N_B(u_2)$, then $\{u_1v_1, u_2v_2\} \subseteq E(G^c)$ and $\{u_1v_2, u_2v_1\} \subseteq E(G)$, by Claim 3, we have $u_1u_2 \in E(G)$, a contradiction. Hence $N_B(u_1) \supseteq N_B(u_2)$. If $|N_B(u_1) \setminus N_B(u_2)| \geq 2$, then there are at least two quasi-edge of the missing edge of u_1u_2 , a contradiction. Therefore, $N_B(u_1) = N_B(u_2) \cup \{y\}$. Similarly, we can prove that $N_A(v_1) = N_A(v_2) \cup \{x\}$, if v_1v_2 is a missing edge in B .

In the graph formed by the missing edges in A , one part X is the vertices of degree odd in B , and the other part Y is the vertices of degree even in B . For any missing edge uv , $\deg_B(u)$ and $\deg_B(v)$ differ by exactly one, so one is odd and the other is even, and hence uv has one end in X and the other in Y , then the graph is bipartite. Similarly, the graph formed by the missing edges in B is a bipartite graph. \square

Claim 5. There exists no vertex in B which dominates A .

Proof. Suppose that there exists a vertex v in B which dominates A . Let $A = \{u_1, u_2, \dots, u_k\}$ and $\text{un}(u_i v) = v_i$ for $i \in \{1, 2, \dots, k\}$. There are $|A|$ edges in $[A, \{v\}]$, then there are $|A|$ missing edges which are incident with v in $B \cup C$. Consider $G + u_0 v$, since $\{u_0, v\}$ does not dominate G , there exists a vertex z such that $u_0 z \mapsto v$ or $v z \mapsto u_0$. If $u_0 z \mapsto v$, then $u_0 z \in E(G)$ and $z \in A$, but $\{u_0, z\}$ does not dominate C , a contradiction. We may assume that $v z \mapsto u_0$. Since $z u_0 \notin E(G)$, $z \in B \cup C$ and $v z \in E(G)$. Then $B \cup C = \{v_1, v_2, \dots, v_k, v, z\}$ by Claim 2 and z dominates $\{v_1, v_2, \dots, v_k\}$. If $k = 1$, then G is a path of length four, it is not a $3\gamma_t$ -edge critical graph, a contradiction. So $k \geq 2$. Since the quasi-edge of $v_i v$ is $u_i v$, u_i dominates $\{v_1, v_2, \dots, v_k\} \setminus \{v_i\}$, where $i \in \{1, 2, \dots, k\}$, then $\{v_1, v_2, \dots, v_k\} \subseteq B$. Moreover, $B = \{v_1, v_2, \dots, v_k, v\}$ and $C = \{z\} = \{v_0\}$. For $1 \leq i < j \leq k$, we have $\{u_i v_i, u_j v_j\} \subseteq E(G^c)$ and $\{u_i v_j, u_j v_i\} \subseteq E(G)$, by Claim 3, $\{u_i u_j, v_i v_j\} \subseteq E(G)$, hence A is a clique and $B \setminus \{v\}$ is also a clique. There are k missing edges in $B \cup C$, but $e_G(A, B) = k(k+1) - k > k$ ($k > 1$), a contradiction. \square

Claim 6. If $|C| \geq 2$, then $[B, C]$ is full.

Proof. Let xy be a missing edge in $[B, C]$, where $x \in B$ and $y \in C$. Consider $G + u_0 y$. Since $\{u_0, y\}$ does not dominate x , there exists a vertex z such that $u_0 z \mapsto y$ or $z y \mapsto u_0$. If $u_0 z \mapsto y$, then $u_0 z \in E(G)$ and $z \in A$, but $\{u_0, z\}$ does not dominate $C \setminus \{y\}$, a contradiction. We may assume that $z y \mapsto u_0$. Since $z y \in E(G)$, $z \in B$ and z dominates A , which contradicts Claim 5. \square

Claim 7. If $|C| \geq 2$, then A is a clique.

Proof. Suppose to the contrary that xy is a missing edge in A . Consider $G + xv_0$. Neither x nor v_0 dominate y , then there exists a vertex z such that $xz \mapsto v_0$ or $zv_0 \mapsto x$. If $zv_0 \mapsto x$, then $zv_0 \in E(G)$ and $\{z, v_0\}$ does not dominate u_0 , a contradiction. We may assume that $xz \mapsto v_0$, then $zv_0 \notin E(G)$, by Claim 1 and 6, $z \in \{u_0\} \cup A$, but $\{x, z\}$ does not dominate $C \setminus \{v_0\}$, a contradiction. \square

Claim 8. $|C| = 1$

Proof. Suppose that $|C| \geq 2$. If B is a clique, then $B \cup C$ and A are all cliques by Claim 1, 6 and 7, consequently, $e_G(A, B) = 0$ by Claim 2 and G is disconnected, a contradiction. We may assume that B is not a clique. Let xy be a missing edge in B . Consider $G + u_0 x$. Neither u_0 nor x dominates y , then there exists a vertex z such that $u_0 z \mapsto x$ or $zx \mapsto u_0$. If $u_0 z \mapsto x$, then $u_0 z \in E(G)$ and $z \in A$, but $\{u_0, z\}$ does not dominate C , a contradiction. We may assume that $zx \mapsto u_0$. Since $u_0 z \notin E(G)$, $z \in B \cup C$, indeed $z \in B$; otherwise, $z \in C$ and x dominates A , which contradicts Claim 5. Since $\{z, x\} \subseteq B$ dominates A , $e_G(A, \{z, x\}) \geq |A|$. By Claim 7, $A \cup \{u_0\}$ is a clique, for any edge e in $[A, \{z, x\}]$, $\text{un}(e) \in B \cup C$, and thus $\text{un}(e) \in B \setminus \{z, x\}$ since $[B, C]$ is full and $zx \in E(G)$. But $|B \setminus \{z, x\}| < |A|$, therefore, there exists $\{e, e'\} \in [A, \{z, x\}]$ such that $\text{un}(e) = \text{un}(e') = w \in B \setminus \{x, z\}$. By Claim 2, e and e' has no common end in B , hence $\{xw, zw\} \in E(G^c)$, which contradicts the fact that $\{x, z\}$ totally dominates $G - u_0$. \square

Claim 9. No vertex in A dominates B .

Proof. Suppose that $u \in A$ dominates B . Hence, for every edge $e \in [u, N_B(v_0)]$, $\text{un}(e) \in A$, and for different edge $e, e' \in [u, N_B(v_0)]$, $\text{un}(e) \neq \text{un}(e')$. Therefore, $|A| \geq |\{u\} \cup \{\text{un}(e) \mid e \in [u, N_B(v_0)]\}| \geq 1 + \deg_G(v_0) \geq 1 + \deg_G(u_0)$, a contradiction. \square

Claim 10. $N_B(v_0) = B$.

Proof. Suppose that $N_B(v_0) \subsetneq B$. Consider $G + u_0v_0$. Since $\{u_0, v_0\}$ does not dominate G , there exists a vertex z such that $u_0z \mapsto v_0$ or $zv_0 \mapsto u_0$. If $u_0z \mapsto v_0$, then $u_0z \in E(G)$, $z \in A$ and z dominates B , which contradicts Claim 9. If $zv_0 \mapsto u_0$, then $zv_0 \in E(G)$, $z \in B$ and z dominates A , which contradicts Claim 5. \square

Claim 11. (a) There exists a vertex w in A such that w does not dominate A and $\deg_B(w) > \frac{|B|}{2}$. Otherwise, (b) A is a clique and there exists a vertex w in B such that w does not dominate B and $\deg_A(w) > \frac{|A|}{2}$.

Proof. Suppose that A is not a clique, let u_1u_2 be a missing edge in A . By Claim 4, we may assume that $\deg_B(u_1) = \deg_B(u_2) + 1$. If $\deg_B(u_1) > \frac{|B|}{2}$, then we are done by taking $w = u_1$. Then we may assume that $\deg_B(u_1) \leq \frac{|B|}{2}$, i.e., $\deg_B(u_1) \leq \left\lfloor \frac{|B|}{2} \right\rfloor$, and thus $\deg_B(u_2) \leq \left\lfloor \frac{|B|}{2} \right\rfloor - 1$.

Consider $G + u_2v_0$. Neither u_2 nor v_0 dominate u_1 , then there exists a vertex z such that $zv_0 \mapsto u_2$ or $u_2z \mapsto v_0$. If $zv_0 \mapsto u_2$, then $zv_0 \in E(G)$ and $z \in B$, but $\{z, v_0\}$ does not dominate u_0 , a contradiction. So we have $u_2z \mapsto v_0$, then $zv_0 \notin E(G)$ and $z \in A$ by Claim 9. Since $\deg_B(u_2) \leq \left\lfloor \frac{|B|}{2} \right\rfloor - 1$, $\deg_B(z) \geq \left\lceil \frac{|B|}{2} \right\rceil + 1 > \frac{|B|}{2}$. If z does not dominate A , then we are done by taking $w = z$. Hence, we may assume that z dominates A . Let $\{y\} = N_B(u_1) \setminus N_B(u_2)$. Since $yu_2 \notin E(G)$, $yz \in E(G)$. Let $x = \text{un}(yz)$. Since $\{y, z\}$ dominates $\{u_0, v_0\} \cup A$, we have $x \in B$, then $\{yu_2, xz, xy\} \subseteq E(G^c)$ and $\{yz, xu_2\} \subseteq E(G)$ (Since $\{u_2, z\}$ dominates $G - v_0$, $xu_2 \in E(G)$), which contradicts Claim 3.

Then we may assume that A is a clique. Similarly, we can prove that there exists a vertex w in B such that w does not dominate B and $\deg_A(w) > \frac{|A|}{2}$. \square

Let $\{U, W\} = \{A, B\}$. By Claim 11, we may assume that there exists a vertex w in W such that w does not dominate W and $\deg_U(w) > \frac{|U|}{2}$. Without loss of generality, among all such vertices in W , we may assume that w is chosen such that $\deg_U(w)$ is maximum.

Claim 12. For every edge e in $[\{w\}, N_U(w)]$, we have $\text{un}(e) \in W$.

Proof. Otherwise, assume that $wv \in [\{w\}, N_U(w)]$ and $\text{un}(wv) = y \in U$. Let wx be a missing edge in W . Since $wv \mapsto y$, $xv \in E(G)$. By Claim 4, we have $N_W(v) = N_W(y) \cup \{w\}$ and hence $xy \in E(G)$. Now, we have $xy \in E(G)$ and $wy \notin E(G)$, i.e., $y \in N_U(x) \setminus N_U(w)$, by Claim 4 again, we have $N_U(x) = N_U(w) \cup \{y\}$, which contradicts the fact that $\deg_U(w)$ is maximum among all the vertices in W satisfying Claim 11. \square

Let $N_U(w) = U_1 = \{v_1, v_2, \dots, v_\ell\}$ and let $w_i = \text{un}(wv_i)$, where $i = 1, 2, \dots, \ell$. Then $w_i \neq w_j$ for $1 \leq i < j \leq \ell$; otherwise, both wv_i and wv_j are quasi-edges of wv_i , which contradicts Claim 2. Let $W_1 = \{w_1, w_2, \dots, w_\ell\}$. By Claim 4, we have $N_U(w) = N_U(w_i) \cup \{v_i\}$, and by Claim 3, W_1 and U_1 are all cliques. Moreover, every vertex v_i dominates U for $i = 1, 2, \dots, \ell$. Hence, for every edge e in $[\{w_1\}, N_U(w_1)]$, we have $\text{un}(e) \in W \setminus (W_1 \cup \{w\})$. Therefore,

$$|W| \geq |W_1 \cup \{w\}| + |N_U(w_1)| = \ell + 1 + (\ell - 1) = 2\ell > |U|. \quad (2.2)$$

If $W = A$ and $U = B$, then $|A| > |B|$, a contradiction. Then $W = B$ and $U = A$. From (2.2) and the fact that $|B| = |A|$ or $|A| + 1$, we conclude that $|B| = |A| + 1 = 2\ell$. For every edge in $[A, B]$, it is the quasi-edge of a missing edge in B since A is a clique by Claim 11. There are at least $\ell^2 + \ell - 1$ edges in $[A, B]$, and there are at most $(2\ell)^2/4$ missing edges in B by Claim 4. Therefore, $\ell = 1$, but it contradicts Claim 5. \square

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